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A Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind with Periodic, Semiperiodic, or Nonperiodic Kernels

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NOTATION

P	projection operator from X onto $\tilde{X}$
X	Banach space of periodic, continuous functions on $[0,1] \times [0,1]$ for the first case and similar definitions for the other cases
$\tilde{X}$	piecewise linear subspace of X
$\bar{X}$	Banach space isomorphic to $\tilde{X}$
Δ_{if}	$\equiv \Delta_i \Delta_f$; $\lvert\Delta_{if}\rvert = \lvert\Delta_i\rvert \lvert\Delta_f\rvert$; if $\lvert\Delta\rvert = \lvert\Delta_{if}\rvert \forall_{ij}$, then $\lvert\Delta_{if}\rvert = \lvert\Delta^2\rvert$
Δ_{ij}	$\equiv$ ijth rectangle of the domain
ϕ	linear extension of ϕ_0 to X; $\phi \equiv \phi_0 P$
ϕ_0	a mapping creating an isomorphism between $\tilde{X}$ and $\bar{X}$
$\omega_s(\delta)$	modulus of continuity of the kernel function $h(s,t)$ relative to s; $\omega_s(\delta) = \sup \lvert h(s + \sigma,t) - h(s,t)\rvert (s \geq 0,\ t \leq 1,\ \lvert\sigma\rvert \leq \delta)$
$\forall$	for all
$\exists$	there exists
$\lVert \ \rVert$	norm in the appropriate space
$\ni$	such that
$\Rightarrow$	implies

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A SOLUTION FOR TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND WITH PERIODIC, SEMIPERIODIC, OR NONPERIODIC KERNELS

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SUMMARY

The convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind was proved in a previous NASA report. This report is an extension of that scheme to two-dimensional Fredholm integral equations of the second kind. The proof of the convergence of the numerical scheme is shown for three cases: the case of periodic kernels, the case of semiperiodic kernels, and the case of nonperiodic kernels.

INTRODUCTION

In reference 1 it is shown that the two-dimensional, incompressible, stationary Navier-Stokes problem can, in general, be represented by a sequence of Fredholm integral equations of the second kind rather than by the traditional nonlinear partial differential equation. Consequently, the task of solving a two-dimensional incompressible stationary Navier-Stokes problem is equivalent to solving this sequence of integral equations.

Accordingly, in this paper an algorithm is developed for numerically solving two-dimensional integral equations. This work is based on that of reference 2, which proves the convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind. In this report the proof of the convergence of the numerical scheme is shown for three cases: (1) the case of periodic kernels; (2) the case of semiperiodic kernels, and (3) the case of nonperiodic kernels.

PERIODIC KERNELS

For the equation

$$x(r,s) - \lambda \int_0^1\int_0^1 h(r,s;t_1,t_2)x(t_1,t_2)dt_1\, dt_2 = y(r,s) \tag{1}$$

it will be demonstrated that an explicit system of equations of the form

$$x(t_{ij}) - \lambda \sum_{k=1}^{n} \sum_{\ell=1}^{n} h(t_{ij}, t_{k\ell}) x(t_{k\ell}) |\Delta^2| = y(t_{ij}) \tag{2}$$

converges to the exact solution of (1), where $t_{ij} = (t_i, t_j)$, and $|\Delta^2_{ij}| = |\Delta_i||\Delta_j|$, and where $h(r,s;t_1,t_2)$ is a continuous function periodic in both variables over the unit square $[0,1] \times [0,1]$, and $y(r,s)$ is also a continuous, periodic function of both variables over the unit square $[0,1] \times [0,1]$. This result will then be extended first to the case of semiperiodic (periodic in one variable only) kernels and then to the case of nonperiodic kernels.

Notice that the following integral

$$\int_0^1 \int_0^1 h(r,s;t_1,t_2) x(t_1,t_2) dt_1 \, dt_2$$

can be approximated by the quadrature formula,

$$\sum_{j=1}^{n} \sum_{i=1}^{n} h(r,s;t_{ij}) x(t_{ij}) |\Delta_i||\Delta_j|$$

and hence equation (2) follows.

Equation (1) will be considered as a functional equation in the space $X = \tilde{C}$ of continuous, periodic functions on $([0,1] \times [0,1])$ the unit square and will typically be expressed in the form

$$Kx \equiv x - \lambda Hx = y \tag{3}$$

The system (2) will be regarded as an approximate functional equation in the space $\bar{X} = R_n$, and will typically be expressed in the form

$$\bar{K}\bar{x} \equiv \bar{x} - \lambda \bar{H}\bar{x} = \phi y \tag{4}$$

Note that we are seeking an exact solution x^* of

$$Kx \equiv x - \lambda Hx = y$$

in the Banach space X and an approximate solution $\tilde{x}^*$ of

$$\tilde{K}\tilde{x} \equiv \tilde{x} - \lambda \tilde{H}\tilde{x} = P_y$$

in the Banach space $\tilde{X} \subseteq X$ where $\tilde{X}$ is a piecewise linear subspace of X and P is a projection operator from X onto $\tilde{X}$. Let ϕ_0 define an isomorphism between X and the space $\bar{X}$ (i.e., $\bar{X}$ is a Banach space (isomorphic to $\tilde{X}$)), and let ϕ be a linear extension of the operation ϕ_0 to the space X. The definition of the linear extension in terms of the projection operator

from X to $\tilde{X}$ and the isomorphism ϕ_0 follows as $\phi \equiv \phi_0 P$. Then $P = \phi_0^{-1}\phi$ and

$$\tilde{K}\tilde{x} = \tilde{x} - \lambda\tilde{H}\tilde{x} = \phi_0^{-1}\phi y$$

If one applies ϕ_0 on the last equation

$$\phi_0(\tilde{x} - \lambda\tilde{H}\tilde{x}) = \phi_0\phi_0^{-1}\phi y$$

$$\phi_0\tilde{x} - \lambda\phi_0\tilde{H}\tilde{x} = \phi_0\phi_0^{-1}\phi y$$

and letting $\bar{x} \equiv \phi_0\tilde{x}$, then the associated equation (4) is obtained as

$$\bar{x} - \lambda\phi_0\tilde{H}\phi_0^{-1}\phi_0\tilde{x} = \phi y$$

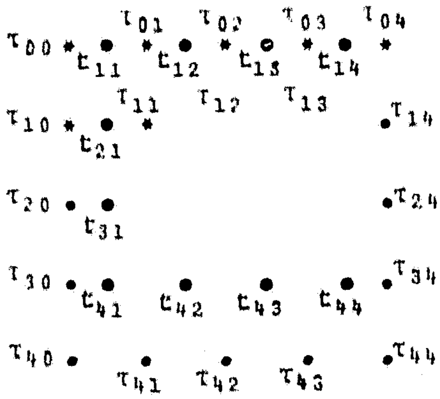
$$\bar{x} - \lambda\phi_0\tilde{H}\phi_0^{-1}\bar{x} = \phi y$$

$$\bar{x} - \lambda\bar{H}\bar{x} = \phi y$$

where

$$\bar{H} \equiv \phi_0\tilde{H}\phi_0^{-1}$$

Divide the unit square, $[0,1] \times [0,1]$, into n^2 smaller squares of area $|\Delta| = 1/n^2$, as shown in the sketch below.



$(n = 4)$

The corners of the squares are denoted by τ_{ij}, t_{ij} being the midpoint on the interval $[\tau_{i-1j-1}, \tau_{i-1,j}]$.

$\bar{H}$ is the $n^2 \times n^2$ matrix,

$$\bar{H} = |\Delta^2| \begin{bmatrix} h(t_{11},t_{11}) & h(t_{11},t_{12}) & \cdots & h(t_{11},t_{1n}) & \cdots & h(t_{11},t_{nn}) \\ h(t_{12},t_{11}) & h(t_{12},t_{12}) & \cdots & h(t_{12},t_{1n}) & \cdots & h(t_{12},t_{nn}) \\ \vdots & & & & & \\ h(t_{1n},t_{11}) & h(t_{1n},t_{12}) & \cdots & h(t_{1n},t_{1n}) & \cdots & h(t_{1n},t_{nn}) \\ h(t_{2n},t_{11}) & h(t_{2n},t_{12}) & \cdots & h(t_{2n},t_{1n}) & \cdots & h(t_{2n},t_{nn}) \\ \vdots & & & & & \\ h(t_{nn},t_{11}) & h(t_{nn},t_{12}) & \cdots & h(t_{nn},t_{1n}) & \cdots & h(t_{nn},t_{nn}) \end{bmatrix}$$

And the transpose of (ϕy) is

$$(\phi y)^T = [y(t_{11}),\ y(t_{12}),\ \ldots\ y(t_{1n}),\ y(t_{21}),\ \ldots,\ y(t_{nn})]$$

Consider the subspace $\tilde{X}_n$ of X, space of continuous, periodic functions that are piecewise linear on the partitions $(\Delta_i)x(\Delta_j)$ of the interval on the unit square, where $|\Delta_i| = [\bar{\tau}_{i+1j},\bar{\tau}_{ij}]$, $|\Delta_j| = [\bar{\tau}_{ij+1},\bar{\tau}_{ij}]$. Define the mapping ϕ_0 of $\tilde{X}$ onto $\bar{X}$ as follows: if $\tilde{x} \in \tilde{X}$, then $\phi_0\tilde{x} = \bar{x}$, where the transpose of $\bar{x}$ is

$$\bar{x}^T = (\bar{\xi}_{01},\bar{\xi}_{02},\ \ldots,\ \bar{\xi}_{0n},\bar{\xi}_{11},\ \ldots\ \bar{\xi}_{1n},\bar{\xi}_{21},\ \ldots\ \bar{\xi}_{2n},\ \ldots,$$

$$\bar{\xi}_{n-11}\ \ldots\ \bar{\xi}_{n-1n})$$

$$\bar{\xi}_{ij} = \tilde{x}(\bar{\tau}_{ij})\ , \quad i = 0,\ \ldots,\ n-1\ , \quad j = 1,\ \ldots\ n$$

For convenience the midpoints are referred to by

$$\bar{\tau}_{ij} = \frac{\tau_{ij-1} + \tau_{ij}}{2}\ , \quad i = 0,\ \ldots,\ n-1\ , \quad j = 1,\ \ldots\ n$$

Note that $\bar{\tau}_{ij}$ and t_{ij} are the same. For $x \in X$, and $x \notin \tilde{X}$,

$$(\phi x)^T = (x(\bar{\tau}_{01}),\ x(\bar{\tau}_{02}),\ \ldots,\ x(\bar{\tau}_{0n}),\ x(\bar{\tau}_{11}),\ \ldots,\ x(\bar{\tau}_{n-1,n})$$

Now, to determine the norms $\|\phi_0\|$, $\|\phi_0^{-1}\|$, and $\|\phi\|$, we shall show first that ϕ_0 is a 1:1 mapping of $\tilde{X}$ onto $\bar{X}$.

If $\phi_0 \tilde{x} = \bar{o}$, $[\bar{o} = (o, o, \ldots, o)] \Rightarrow \tilde{x} = o$ ϕ_0 is 1:1. Since ϕ_0 defines a 1:1 mapping between $\tilde{X}$ and R_n, its inverse ϕ_0^{-1} exists:

$$\tilde{x} = \phi_0^{-1} \bar{x}$$

where

$$\bar{x}^T = (\bar{\xi}_{ij}) , \quad i = 0, \ldots n-1 , \quad j = 1, \ldots n$$

$$\bar{\xi}_{ij} = \tilde{x}(\bar{\tau}_{ij})$$

$$\|\phi_0\| = \sup_{\|\tilde{x}\|=1} \|\phi_0 \tilde{x}\| = 1$$

Hence

$$\|\phi_0^{-1}\| = 1, \quad \text{and} \quad \|\phi_0\| = 1$$

Now, within the functional framework as developed in the last several paragraphs, it is possible to apply the general theory developed by Kantorovich (ref. 1) in order to prove that the solutions of the approximate equation (4) converge to the solutions of the exact equation (3) as $n \to \infty$. In essence, this reduces to showing that the three following conditions can be satisfied:

Condition A:

$$\forall \tilde{x} \in \tilde{X} , \quad \|\phi H \tilde{x} - \bar{H} \phi_0 \tilde{x}\| \leq \zeta \|\tilde{x}\|$$

Condition B:

$$\forall x \in X , \quad \exists \tilde{x} \in \tilde{X} \ni \|Hx - \tilde{x}\| \leq \zeta_1 \|x\|$$

Condition C:

$$\exists \tilde{y} \in \tilde{X} \ni \|y - \tilde{y}\| \leq \zeta_2 \|y\|$$

Condition A

First, for condition A, we shall show that for every $\tilde{x} \in \tilde{X}$, $\|\phi H \tilde{x} - \bar{H} \phi_0 \tilde{x}\| \leq \zeta \|\tilde{x}\|$. At the midpoints, we have (for convenience and to avoid confusion, midpoints are denoted by $[t_{ij} = \bar{\tau}_{ij}]$):

$$\phi H \tilde{x} = \left[\int_0^1 \int_0^1 h(\bar{\tau}_{01}, t) \tilde{x}(t) dt, \int_0^1 \int_0^1 h(\bar{\tau}_{02}, t) \tilde{x}(t) dt, \ldots, \right.$$
$$\left. \int_0^1 \int_0^1 h(\bar{\tau}_{n-1,n}, t) \tilde{x}(t) dt \right]$$

$$\phi_0\tilde{x} = [\tilde{x}(\bar{\tau}_{01}),\tilde{x}(\bar{\tau}_{02}),\ \ldots,\tilde{x}(\bar{\tau}_{0n}),\tilde{x}(\bar{\tau}_{11}),\tilde{x}(\bar{\tau}_{12}),\ \ldots,\tilde{x}(\bar{\tau}_{1n}),\ \ldots,$$

$$\tilde{x}(\bar{\tau}_{n-11}),\ \ldots,\tilde{x}(\bar{\tau}_{n-1n})]$$

$$\bar{H}\phi_0\tilde{x} = |\Delta^2| \left[\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\overline{\tau_{01}},\bar{\tau}_{ij})\tilde{x}(\overline{\tau_{ij}}),\ \sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\overline{\tau_{02}},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}),\ \ldots,\right.$$

$$\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,1},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}),\ \ldots,$$

$$\left.\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,n},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})\right]$$

Therefore,

$$\phi H\tilde{x} - \bar{H}\phi_0\tilde{x} = \left[\int_0^1\int_0^1 h(\bar{\tau}_{01},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{01},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}),\ \ldots,\right.$$

$$\left.\int_0^1\int_0^1 h(\bar{\tau}_{n-1,n},t)\tilde{x}(d)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,n},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})\right]$$

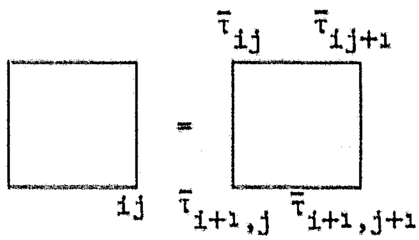
Hence the $(k\ell)$th component is

$$I_{k\ell} = \int_0^1\int_0^1 h(\bar{\tau}_{k\ell},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{k\ell},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})$$

Let $h(\bar{\tau}_{k\ell},t) \equiv z(t)$; then

$$I_{k\ell} = \int_0^1\int_0^1 z(t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} z(\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})$$

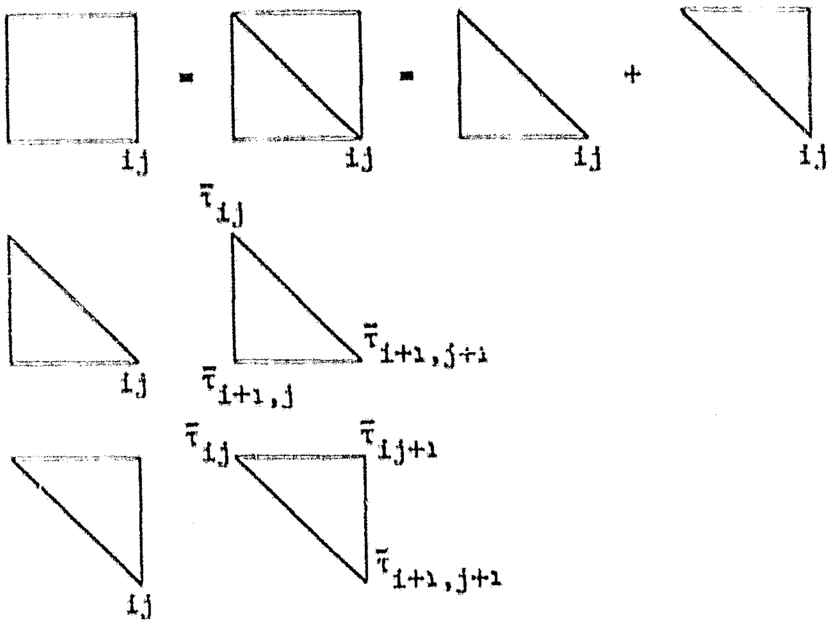
Let us define the limits of integration by the following notation:



Then,

$$I_{k\ell} = \sum_{i=0}^{n-1} \sum_{j=1}^{n} \iint_{\square_{ij}} z(t)\tilde{x}(t) - z(\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})$$

Integration over the small square ij, $\square_{ij}$ can be broken into integrations over two triangles as shown



Therefore,

$$I_{k\ell} = \sum_{i=0}^{n-1} \sum_{j=1}^{n} \left(\iint_{\triangle_{ij}} [\quad] + \iint_{\nabla_{ij}} [\quad] \right)$$

where

$$[\quad] = z(t)\tilde{x}(t) - z(\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})$$

$$\iint\limits_{\triangle_{ij}} [\quad] = \iint\limits_{\triangle_{ij}} [z(t) - z(\bar{\tau}_{ij})]\tilde{x}(t)dt + \iint\limits_{\triangle_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt$$

And, similarly,

$$\iint\limits_{\bigtriangledown_{ij}} [\quad] = \iint\limits_{\bigtriangledown_{ij}} [z(t) - z(\bar{\tau}_{ij})]\tilde{x}(t)dt + \iint\limits_{\bigtriangledown_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt$$

And,

$$\iint\limits_{\square_{ij}} [\quad] = \iint\limits_{\square_{ij}} [z(t) - z(\bar{\tau}_{ij})]\tilde{x}(t)dt + \iint\limits_{\square_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt$$

$$\iint\limits_{\triangle_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - x(\bar{\tau}_{ij})]dt = \frac{\Delta^2}{2} z(\bar{\tau}_{ij})[\tilde{x}(\bar{t}_{ij}) - \tilde{x}(\bar{\tau}_{ij})]$$

$$\iint\limits_{\bigtriangledown_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})dt = \frac{\Delta^2}{2} z(\bar{\tau}_{ij})[\tilde{x}(\bar{t}^{ij}) - \tilde{x}(\bar{\tau}_{ij})]$$

where

$$\tilde{x}(\bar{t}^{ij}) = \frac{\tilde{x}(\bar{\tau}_{ij}) + \tilde{x}(\bar{\tau}_{ij+1}) + \tilde{x}(\bar{\tau}_{i+1,j+1})}{3}$$

$$\tilde{x}(\bar{t}_{ij}) = \frac{\tilde{x}(\bar{\tau}_{ij}) + \tilde{x}(\bar{\tau}_{i+1,j}) + \tilde{x}(\bar{\tau}_{i+1,j+1})}{3}$$

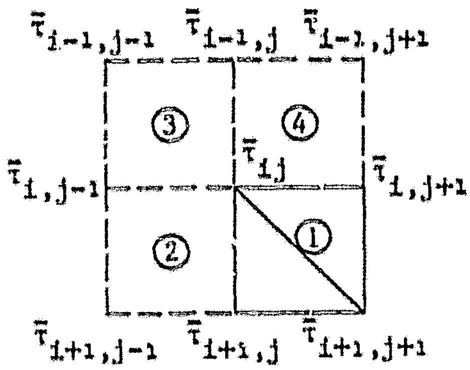
$$\iint\limits_{\square_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt$$

$$= \frac{\Delta^2}{6} z(\bar{\tau}_{ij})[2\tilde{x}(\bar{\tau}_{i+1,j+1}) + \tilde{x}(\bar{\tau}_{i,j+1}) + \tilde{x}(\bar{\tau}_{i+1,j}) - 4\tilde{x}(\bar{\tau}_{ij})]$$

Estimates for

$$\sum_{i=1}^{n}\sum_{j=2}^{n+1}\iint_{\square_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt \equiv \sum\sum \bar{I}_{ij}$$

will be computed next. Consider an arbitrary interior square, $\square_{ij}$, the squares that contribute as functions of $\bar{\tau}_{ij}$ are ①, ②, ③, and ④:



Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \bar{I}_{ij} = {} & \frac{\Delta^2}{6} z(\bar{\tau}_{ij})[2\tilde{x}(\bar{\tau}_{i+1,j+1}) + \tilde{x}(\bar{\tau}_{i,j+1}) + \tilde{x}(\bar{\tau}_{i+1,j}) - 4\tilde{x}(\bar{\tau}_{ij})] \\ & + \frac{\Delta^2}{6} z(\bar{\tau}_{i,j-1})[2\tilde{x}(\bar{\tau}_{i+1,j}) + \tilde{x}(\bar{\tau}_{ij}) + \tilde{x}(\bar{\tau}_{i+1,j-1}) - 4\tilde{x}(\bar{\tau}_{i,j-1})] \\ & + \frac{\Delta^2}{6} z(\bar{\tau}_{i-1,j-1})[2\tilde{x}(\bar{\tau}_{ij}) + \tilde{x}(\bar{\tau}_{i-1,j}) + \tilde{x}(\bar{\tau}_{i,j-1}) - 4\tilde{x}(\bar{\tau}_{i-1,j-1})] \\ & + \frac{\Delta^2}{6} z(\bar{\tau}_{i-1,j})[2\tilde{x}(\bar{\tau}_{i,j+1}) + \tilde{x}(\bar{\tau}_{i-1,j+1}) + \tilde{x}(\bar{\tau}_{ij}) - 4\tilde{x}(\bar{\tau}_{i-1,j})] \\ & + \ldots .\end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \bar{I}_{ij} = \frac{\Delta^2}{6}\tilde{x}(\bar{\tau}_{ij})[z(\bar{\tau}_{i,j-1}) - 4z(\bar{\tau}_{ij}) + 2z(\bar{\tau}_{i-1,j-1}) + z(\bar{\tau}_{i-1,j})] + \ldots$$

Terms will be similar to those in this expression for $\bar{\tau}_{ij}$ for all squares $\square_{ij}$ for $i = 1, \ldots, n-1$; $j = 1, \ldots, n-1$. Therefore, it remains to obtain estimates for rectangles $\square_{ij}$, $i = n$ with $j = 1, \ldots, n-1$ and for $j = n$, with $i = 1, \ldots, n-1$, and, also for $i = n$, $j = n$.

Now, consider $i = n$, $j = n$. We shall have terms from the rectangles $\square_{nn}$, $\square_{n1}$, $\square_{1n}$, $\square_{11}$

$$\sum\sum \bar{I}_{ij} = \frac{\Delta^2}{6} z(t_{nn})[\tilde{x}(\tau_{n-1,n-1}) + \tilde{x}(\tau_{nn}) + 2\tilde{x}(\tau_{n,n-1}) - 4\tilde{x}(\tau_{n-1,n})]$$

$$+ \frac{\Delta^2}{6} z(t_{n1})[\tilde{x}(\tau_{n-1,0}) + \tilde{x}(\tau_{n1}) + 2\tilde{x}(\tau_{n0}) - 4\tilde{x}(\tau_{n-1,1})]$$

$$+ \frac{\Delta^2}{6} z(t_{1n})[\tilde{x}(\tau_{0,n-1}) + \tilde{x}(\tau_{1n}) + 2\tilde{x}(\tau_{1,n-1}) - 4\tilde{x}(\tau_{0n})]$$

$$+ \frac{\Delta^2}{6} z(t_{11})[\tilde{x}(\tau_{0,0}) + \tilde{x}(\tau_{11}) + 2\tilde{x}(\tau_{1,0}) - 4\tilde{x}(\tau_{0,1})]$$

Since

$$\tilde{x}(\tau_{n0}) = \tilde{x}(\tau_{nn})$$

$$\tilde{x}(\tau_{0n}) = \tilde{x}(\tau_{nn})$$

$$\tilde{x}(\tau_{0,0}) = \tilde{x}(\tau_{nn})$$

$$\sum\sum \bar{I}_{ij} = \frac{\Delta^2}{6} \tilde{x}(\tau_{nn})[z(t_{nn}) + 2z(t_{n1}) - 4z(t_{1n}) + z(t_{11})]$$

However,

$$z(t_{n1}) = z(t_{n,n+1})$$

$$z(t_{1n}) = z(_{n+1,n})$$

Therefore, for $\square_{nn}$ (τ_{nn}), it yields

$$\frac{\Delta^2}{6} \tilde{x}(\tau_{nn})[z(t_{n,n}) + 2z(t_{n+2,1}) + z(t_{n+1,n+1}) - 4z(t_{n+1,n})] \leq \frac{\Delta^2}{6} \|\tilde{x}\|[4\omega_z(\sqrt{2}\Delta)]$$

Hence, it directly follows that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} \int_{\square_{ij}} z(t_{ij})[\tilde{x}(t) - \tilde{x}(\tau_{i-1,j})]dt\right| \leq \frac{\Delta^2 n^2}{6} 4\omega_z(\sqrt{2}\Delta)\|\tilde{x}\| = \frac{2n^2}{3} \omega_z(\sqrt{2}\Delta^3)\|x\|$$

Therefore,

$$I_{k\ell} = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\square_{ij}} [z(t)\tilde{x}(t) - z(t_{ij})\tilde{x}(\tau_{i-1,j})]dt$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\square_{ij}} [z(t) - z(t_{ij})]\tilde{x}(t)dt + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\square_{ij}} z(t_{ij})[\tilde{x}(t) - x(\tau_{i-1,j})]dt$$

$$\left|I_{k\ell}\right| \le \omega_z\left(\frac{3}{4}\,\Delta\right)\|\tilde{x}\| + \frac{2}{3}\,\omega_z(\sqrt{2}\Delta)\|\tilde{x}\|$$

for all k and ℓ.

$$\left|I_{k\ell}\right| \le \zeta\|\tilde{x}\| \,, \quad \text{for} \quad \zeta = \omega_z\left(\frac{3}{4}\,\Delta\right) + \frac{2}{3}\,\omega_z(\sqrt{2}\Delta)$$

Hence,

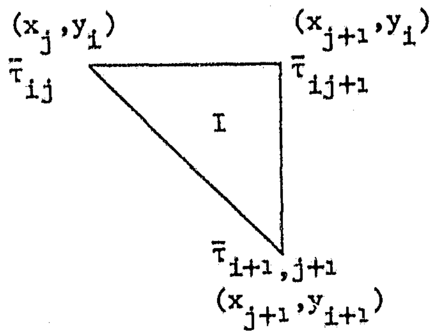
$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \zeta\|\tilde{x}\|$$

Condition B

Condition B has been defined as

$$\forall x \in X \; \exists \; \tilde{x} \in \tilde{X} \ni \|Hx - \tilde{x}\| \le \zeta_1\|x\|$$

Suppose, for the sake of discussion, that $(x,y) \in I$, as described by the sketch below for triangle I,



Let

$$\tilde{z} = \frac{1}{\Delta}\left\{[(x - x_j) - (y - y_i)]z_{i,j+1} + z_{ij}(x_{j+1} - x) + z_{i+1,j+1}(y - y_i)\right\}$$

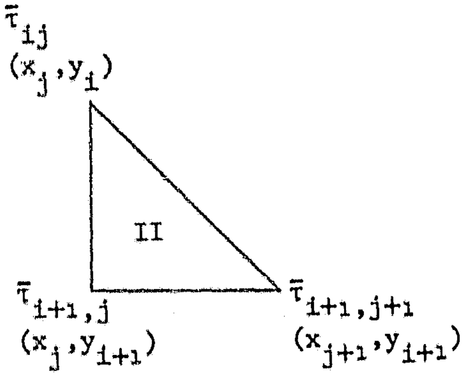
$$z = \frac{1}{\Delta}\left\{[(x - x_j) - (y - y_i)]z + z(x_{j+1} - x) + (y - y_i)z\right\}$$

$$z - \tilde{z} = \frac{1}{\Delta}\left\{[(x - x_j) - (y - y_i)](z - z_{ij+1}) + (z - z_{ij})(x_{j+1} - x) + (z - z_{i+1,j+1})(y - y_i)\right\}$$

$$|z - \tilde{z}| \le \frac{1}{\Delta}\,[|(x - x_j) - (y - y_i)|\omega_z(\sqrt{2}\Delta) + \omega_z(\sqrt{2}\Delta)|x_{j+1} - x| + \omega_z(\sqrt{2}\Delta)|y - y_i|]$$

$$\le 2\omega_z(\sqrt{2}\Delta)$$

Similarly for the triangle II:



$$|z - \tilde{z}| \le 2\omega_z(\sqrt{2}\Delta)$$

$$H\bar{w} = \int_0^1 \int_0^1 h(r,s,t_1,t_2)\bar{w}(t_1,t_2)dt$$

Now, assume that $(r,s) \in \square_{ij}$. Suppose $(r,s) \in \square_{ij}$, and let

$$z = H\bar{w} \equiv \int_0^1 \int_0^1 h(r,s;t_1,t_2)\bar{w}(t_1,t_2)dt_1\, dt_2$$

$$z - \tilde{z} = \frac{1}{\Delta}\Big\{[(x - x_{j-1}) - (y_i - y)] \int_0^1 \int_0^1 [h(r,s;t_1,t_2)$$

$$- h(x_j,x_i;t_1,t_2)]\bar{w}(t_1,t_2)dt_1\, dt_2$$

$$+ (x_j - x) \int_0^1 \int_0^1 [h(r,s;t_1,t_2) - h(x_{j-1},y_i)]\bar{w}(t_1,t_2)dt_1\, dt_2$$

$$+ (y_i - y) \int_0^1 \int_0^1 [h(r,s;t_1,t_2) - h(x_j,y_{i-1})]\bar{w}(t_1,t_2)dt_1\, dt_2\Big\}$$

$$|z - \tilde{z}| \le \frac{1}{\Delta}\,[(x - x_{j-1})\omega(\sqrt{2}\Delta)\|\bar{w}\| + (x_j - x)\omega(\sqrt{2}\Delta)\|\bar{w}\| + (y_i - y)\omega(\sqrt{2}\Delta)\|\bar{w}\|]$$

$$\le \frac{\omega(\sqrt{2}\Delta)}{\Delta}\|\bar{w}\|[2\Delta] \le 2\omega(\sqrt{2}\Delta)\|\bar{w}\|$$

Similarly for $(r,s) \in \nabla_{ij}$. Hence $\|Hx - \tilde{x}\| \le \zeta_1\|x\|$ with $\zeta_1 = 2\omega(\sqrt{2}\Delta)$.

Condition C

It remains to show the third condition which is defined as follows. For each element $y \in X$, there exists a $\tilde{y} \in \tilde{X}$ such that

$$\|y - \tilde{y}\| \le \zeta_2\|y\|$$

From condition B, we have that

$$\|y - \tilde{y}\| \le 2\omega(\sqrt{2}\Delta)$$

(ω is the modulus of continuity for the function h.) Therefore, letting

$$\zeta_2 = \frac{2\omega(\sqrt{2}\Delta)}{\|y\|}$$

we have

$$\|y - \tilde{y}\| \le \zeta_2\|y\|$$

which completes the proof.

SEMIPERIODIC KERNELS

As in the preceding discussion on periodic kernels, a semiperiodic function of two variables is defined as a function of two variables that is periodic in at least one variable. To prove that the solutions of the approximate equation (4) converge to the solution of the exact equation (3) as $n \to \infty$, we must show that the following three conditions are satisfied (ref. 1).

Condition A:

$$\forall \tilde{x} \in \tilde{X}\ , \quad \|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \leq \zeta\|\tilde{x}\|$$

Condition B:

$$\forall x \in X\ , \quad \exists\, \tilde{x} \in \tilde{X} \ni \|Hx - \tilde{x}\| \leq \zeta_1\|x\|$$

Condition C:

$$\exists\, \tilde{y} \in \tilde{X} \ni \|y - \tilde{y}\| \leq \zeta_2\|y\|$$

Condition A

To prove condition A $\left(\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \leq \eta\|\tilde{x}\| \text{ for } H\tilde{x} \notin \tilde{X}\right)$, for semiperiodic kernels:

$$\phi H\tilde{x} = \left[\int_0^1 \int_0^1 h(\bar{\tau}_{01},t)\tilde{x}(t)dt, \ldots, \int_0^1 \int_0^1 h(\bar{\tau}_{0n},t)\tilde{x}(t)dt, \ldots,\right.$$

$$\left.\int_0^1 \int_0^1 h(\bar{\tau}_{n-1,1},t)\tilde{x}(t)dt, \ldots, \int_0^1 \int_0^1 h(\bar{\tau}_{n-1,n},t)\tilde{x}(t)dt\right]$$

$$\phi_0\tilde{x} = [x(\bar{\tau}_{01}), \tilde{x}(\bar{\tau}_{02}), \ldots, \tilde{x}(\bar{\tau}_{0n}), \ldots, \tilde{x}(\bar{\tau}_{n-1,1}), \ldots, \tilde{x}(\bar{\tau}_{n-1,n})]$$

$$\bar{H}\phi_0\tilde{x} = \Delta^2\left[\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{01},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}), \ldots, \sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{0n},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}), \ldots,\right.$$

$$\left.\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,1},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}) \ldots, \sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,n},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})\right]$$

$$\phi H\tilde{x} - \bar{H}\phi_0\tilde{x} = \Bigg[\int_0^1\int_0^1 h(\bar{\tau}_{01},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{01},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}), \ldots,$$

$$\int_0^1\int_0^1 h(\bar{\tau}_{0n},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{0n},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}), \ldots,$$

$$\int_0^1\int_0^1 h(\bar{\tau}_{n-1,1},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,1},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij}) \ldots,$$

$$\int_0^1\int_0^1 h(\bar{\tau}_{n-1,n},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{n-1,n},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})\Bigg]$$

Let

$$I_{k\ell} = \int_0^1\int_0^1 h(\bar{\tau}_{k\ell},t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} h(\bar{\tau}_{k\ell},\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})$$

and let $h(\bar{\tau}_{k\ell},t) = z(t)$. Then

$$I_{k\ell} = \int_0^1\int_0^1 z(t)\tilde{x}(t)dt - \Delta^2\sum_{i=0}^{n-1}\sum_{j=1}^{n} z(\bar{\tau}_{ij})x(\bar{\tau}_{ij})$$

$$= \sum_{i=0}^{n-1}\sum_{j=1}^{n}\iint_{\square_{ij}} [z(t)\tilde{x}(t) - z(\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})]dt$$

Since

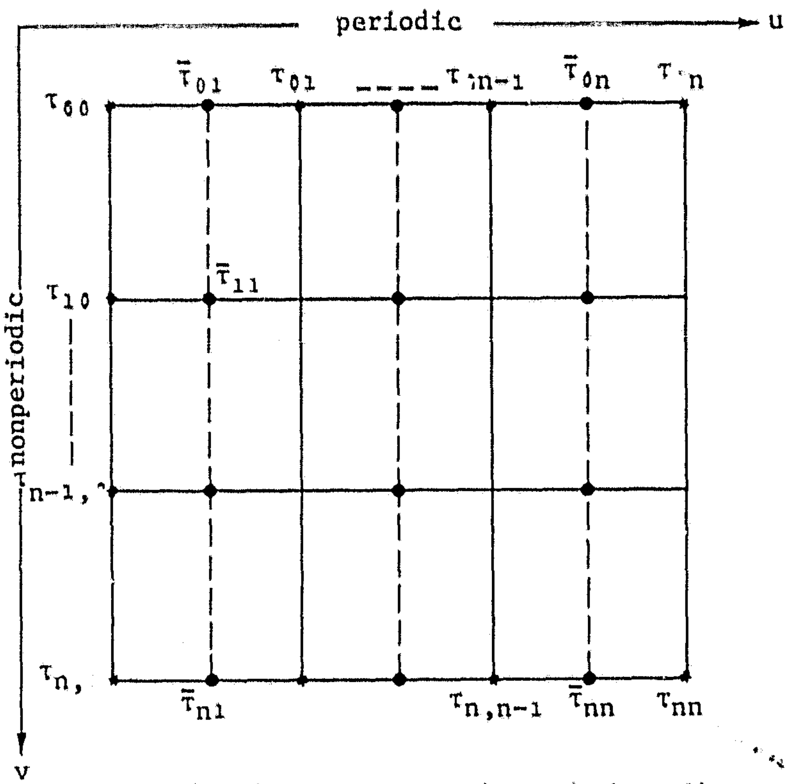
$$\iint_{\square_{ij}} [z(t)\tilde{x}(t) - z(\bar{\tau}_{ij})\tilde{x}(\bar{\tau}_{ij})]dt$$

$$= \int_{\square_{ij}} [z(t) - z(\bar{\tau}_{ij})]\tilde{x}(t)dt + \int_{\square_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt$$

$$\sum_{i=0}^{n-1}\sum_{j=1}^{n}\iint_{\square_{ij}} [z(t) - z(\bar{\tau}_{ij})]\tilde{x}(t)dt \le \omega_z(\Delta)\ z$$

$$\iint_{\square_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt = \frac{\Delta^2}{6} z(\bar{\tau}_{ij})[2\tilde{x}(\bar{\tau}_{i+1,j+1}) + \tilde{x}(\bar{\tau}_{i,j+1}) + \tilde{x}(\bar{\tau}_{i+1,j}) - 4\tilde{x}(\bar{\tau}_{ij})]$$

Semiperiodicity is defined as being periodic along one of the axes (u) and nonperiodic along the other (v).



τ_{ij}	$i = 0, \ldots n$ $j = 0, \ldots n$	$(n + 1)x(n + 1)$	
$\bar{\tau}_{ij}$	$i = 0, \ldots n$ $j = 1, \ldots n$	$nx(n - 1)$ $(n + 1)$ rows n columns	

where $\bar{\tau}_{ij}$ are the midpoints of τ_{ij} as shown in the above sketch.

$\tilde{X} = \begin{cases} x \mid x(u,v) \in \square_{ij}, \text{ on } \triangle_{ij}, \ x \text{ is determined by its values at vertices of} \\ \text{triangle; also for } \nabla_{ij}, \text{ where } \square_{ij} = \triangle_{ij} + \nabla_{ij} \text{ for} \\ i = 0, \ldots, n-2, \ j = 1, \ldots, n; \text{ i.e., in the same manner as for the} \\ \text{periodic case.} \end{cases}$

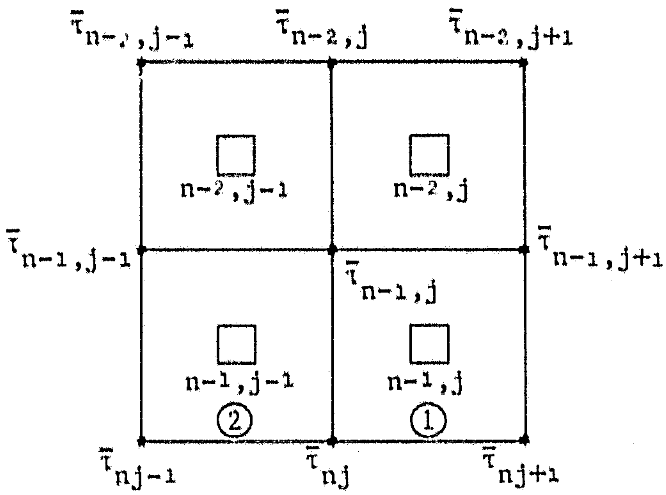
For $(\bar{u},\bar{v})$ lying on the segment $[\bar{\tau}_{n-1,j}, \bar{\tau}_{n-1,j+1}]$

let

$$\tilde{x}(\bar{u},v) = \tilde{x}(\bar{u},\bar{v}) \quad \text{for} \quad v \geq \bar{v} .$$

For midpoints $\bar{\tau}_{ij}$, coefficients for $\tilde{x}(\bar{\tau}_{ij})$ are the same as for the periodic case. Since periodic in the u direction, the boundaries (where v is fixed) are handled in exactly the same way as for the periodic case. Therefore, the only additional contributions arise around the boundaries where u is constant.

Now, let us determine the additional terms arising from the above mentioned items.



The coefficient for $\tilde{x}(\bar{\tau}_{n-1,j})$ comes from the following terms:

$$\int_{\square_{n-2,j-1}} + \int_{\square_{n-2,j}} + \int_{\square_{n-1,j-1}} + \int_{\square_{n-1,j}}$$

$$= \frac{\Delta^2}{6} z(\bar\tau_{n-2,j-1})[-4\tilde{x}(\bar\tau_{n-2,j-1}) + \underline{2\tilde{x}(\bar\tau_{n-1,j})} + \tilde{x}(\bar\tau_{n-1,j-1}) + \tilde{x}(\bar\tau_{n-2,j})]$$

$$+ \frac{\Delta^2}{6} z(\bar\tau_{n-2,j})[-4\tilde{x}(\bar\tau_{n-2,j}) + 2\tilde{x}(\bar\tau_{n-1,j+1}) + \underline{\tilde{x}(\bar\tau_{n-1,j})} + \tilde{x}(\bar\tau_{n-2,j+1})]$$

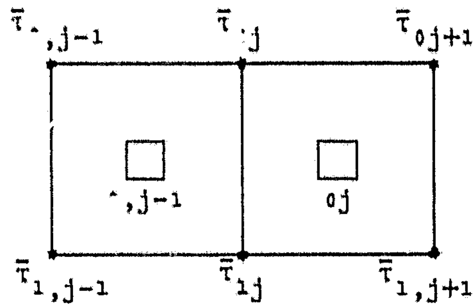
$$+ \frac{\Delta^2}{6} z(\bar\tau_{n-1,j-1})[-4\tilde{x}(\bar\tau_{n-1,j-1}) + 2\tilde{x}(\bar\tau_{nj}) + \tilde{x}(\bar\tau_{n,j-1}) + \underline{\tilde{x}(\bar\tau_{n-1,j})}]$$

$$+ \frac{\Delta^2}{6} z(\bar\tau_{n-1,j})[\underline{-4\tilde{x}(\bar\tau_{n-1,j})} + 2\tilde{x}(\bar\tau_{n,j+1}) + \tilde{x}(\bar\tau_{nj}) + \tilde{x}(\bar\tau_{n-1,j+1})]$$

Underlined terms give the coefficient of $\tilde{x}(\bar\tau_{n-1,j})$ as

$$\frac{\Delta^2}{6} \tilde{x}(\bar\tau_{n-1,j})[2z(\bar\tau_{n-2,j-1}) + z(\bar\tau_{n-2,j}) + z(\bar\tau_{n-1,j-1}) - 4z(\bar\tau_{n-1,j})]$$

But, the coefficient for $x(\bar\tau_{nj})$ arises from two squares only: $\square_{n-1,j-1} + \square_{n-1,j}$.



The coefficient for $\tilde{x}(\bar\tau_{0j})$ comes from the following terms:

$$\int_{\square_{0,j-1}} + \int_{\square_{0,j}} = \frac{\Delta^2}{6} z(\bar\tau_{0,j-1})[2\tilde{x}(\bar\tau_{1j}) + \underline{\tilde{x}(\bar\tau_{0j})} + \tilde{x}(\bar\tau_{1j-1}) - 4\tilde{x}(\bar\tau_{0,j-1})]$$

$$+ \frac{\Delta^2}{6} z(\bar\tau_{0j})[2\tilde{x}(\bar\tau_{1,j+1}) + \tilde{x}(\bar\tau_{1j}) + \tilde{x}(\bar\tau_{0,j+1}) - \underline{4\tilde{x}(\bar\tau_{0j})}]$$

Underlined terms give the coefficient of $\tilde{x}(\bar\tau_{0j})$:

$$\tilde{x}(\bar{\tau}_{0j}) = \frac{\Delta^2}{6}\,\tilde{x}(\bar{\tau}_{0j})[z(\bar{\tau}_{0,j-1}) - 4z(\bar{\tau}_{0j})]$$

Summing over j

$$\frac{\Delta^2}{6}\sum_{j=1}^{n} \tilde{x}(\bar{\tau}_{0j})[z(\bar{\tau}_{0,j-1}) - 4z(\bar{\tau}_{0j})] \le \frac{\Delta^2}{6}\, n\|\tilde{x}\|5\|z\| \le \Delta\|\tilde{x}\|\ \|z\|$$

$$\int_{\square_{n-1,j-1}} + \int_{\square_{n-1,j}} = \frac{\Delta^2}{6}\Big[z(\bar{\tau}_{n-1,j-1})[-4x(\bar{\tau}_{n-1,j-1}) + \underline{2\tilde{x}(\bar{\tau}_{nj})} + \tilde{x}(\bar{\tau}_{n,j-1})$$

$$+ \tilde{x}(\bar{\tau}_{n-1,j})\Big] + \frac{\Delta^2}{6}\, z(\bar{\tau}_{n-1,j})[-4\tilde{x}(\bar{\tau}_{n-1,j})$$

$$+ 2\tilde{x}(\bar{\tau}_{n,j+1}) + \underline{\tilde{x}(\bar{\tau}_{nj})} + \tilde{x}(\bar{\tau}_{n-1,j+1})]$$

Underlined terms give the coefficient of

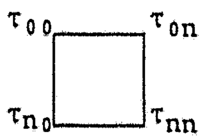
$$\tilde{x}(\bar{\tau}_{nj}) = \frac{\Delta^2}{6}\,\tilde{x}(\bar{\tau}_{nj})[2z(\bar{\tau}_{n-1,j-1}) + z(\bar{\tau}_{n-1,j})]$$

Summing over j

$$\frac{\Delta^2}{6}\sum_{j=1}^{n} \tilde{x}(\bar{\tau}_{nj})[2z(\bar{\tau}_{n-1,j-1}) + z(\bar{\tau}_{n-1,j})]$$

$$\le \frac{\Delta^2}{6}\,\|\tilde{x}\|3\|z\|n \le \frac{\Delta}{2}\,\|\tilde{x}\|\ \|z\|$$

This is the additional term arising from the terms of the last two rows of the square



in the nonperiodic direction. Therefore,

$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \left[\omega_h\!\left(\frac{3}{4}\,\Delta\right) + \frac{2}{3}\,\omega_h(\Delta\sqrt{2})\right]\|\tilde{x}\| + \frac{\Delta}{2}\,\|h\|\ \|\tilde{x}\| + \Delta\|h\|\ \|\tilde{x}\|$$

or

$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \left[\omega_h\left(\frac{3}{4}\Delta\right) + \frac{2}{3}\omega_h(\sqrt{2}\Delta) + \frac{3}{2}\Delta\|h\|\right]\|\tilde{x}\|$$

is shown for the semiperiodic case (ω_h is the modulus of continuity for the function h).

Conditions B and C

Conditions B and C can be shown to be satisfied in exactly the same way as in the preceding section on periodic kernels. This completes the proof.

NONPERIODIC KERNELS

To prove that the solutions of the approximate equation (4) converge to the solution of the exact equation (3) as $n \to \infty$, we must show that the following three conditions can be satisfied (ref. 1).

Condition A:

$$\forall \tilde{x} \in \tilde{X}, \quad \|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \zeta\|\tilde{x}\|$$

Condition B:

$$\forall x \in X, \quad \exists \tilde{x} \in \tilde{X} \ni \|Hx - \tilde{x}\| \le \zeta_1\|x\|$$

Condition C:

$$\exists \tilde{y} \in \tilde{X} \ni \|y - \tilde{y}\| \le \zeta_2\|y\|$$

Condition A

Condition A is

$$\forall \tilde{x} \in \tilde{X}, \quad \|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \zeta\|\tilde{x}\|$$

Definition of ϕ:

$$\phi x = [x(\tau_{00}), x(\bar{\tau}_{01}), \ldots, x(\bar{\tau}_{0n}), x(\tau_{10}), x(\bar{\tau}_{11}), \ldots, x(\bar{\tau}_{1n}) \ldots, x(\bar{\tau}_{n-1,n})]$$

$n(n + 1)$ components are involved.

$$\|\phi\| = \sup_{\|x\|=1} \|\phi x\| = 1$$

Definition of ϕ_0:

$$\phi_0\tilde{x} = [\tilde{x}(\tau_{00}), \tilde{x}(\bar{\tau}_{01}), \ldots, \tilde{x}(\bar{\tau}_{0n}), \tilde{x}(\tau_{10}), \tilde{x}(\bar{\tau}_{11}), \ldots, \tilde{x}(\bar{\tau}_{1n})$$

$$\ldots, \tilde{x}(\bar{\tau}_{n-1,n})]$$

$n(n+1)$ components are involved,

$$\|\phi_0\| = \sup_{\|x\|=1} \|\phi_0\tilde{x}\| = 1$$

$$\phi_0\tilde{x} = \bar{x} \; ; \quad \phi_0^{-1}\bar{x} = \tilde{x}$$

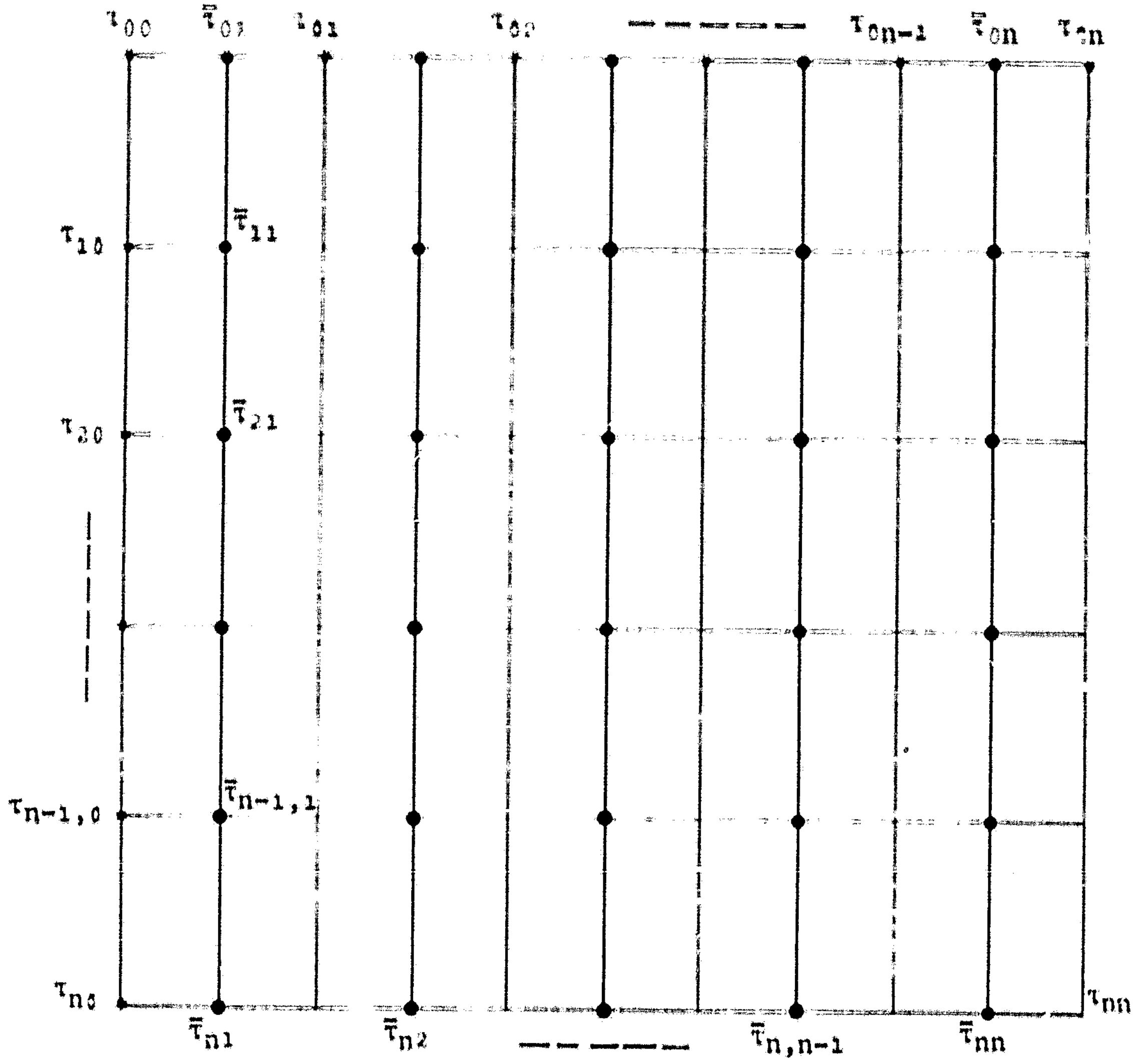
If $\bar{x} = (\xi_{00}, \xi_{01}, \ldots, \xi_{0n}, \xi_{10}, \xi_{11}, \ldots, \xi_{1n}, \ldots, \xi_{n-1,n})$ then

$$\tilde{x}(\bar{\tau}_{ij}) = \xi_{ij} \quad \text{for} \quad j \neq 0$$

$$\tilde{x}(\tau_{i0}) = \xi_{i0} \quad \text{for} \quad j = 0$$

$$\|\phi_0^{-1}\| = \sup_{\|\bar{x}\|=1} \left\{\|\phi_0^{-1}\bar{x}\|\right\} = 1$$

Note that the elements $\tilde{x} \in \tilde{X}$ are constructed in the same manner as the elements were constructed in the semiperiodic case for the interior squares. The set of values (u,v) of $\tilde{x}$, which are left of the line segment connecting τ_{i1}'s ($i = 0, \ldots, n$), are determined by the triangles constructed from the values of the vertices of the rectangles (see diagram on following page), i.e., constructed in the same manner as for the interior squares, and for values $(u,\bar{v}) \geq (\bar{u},\bar{v})$ where $(\bar{u},\bar{v})$ lies on the segment joining $\bar{\bar{\tau}}_{in}$'s, $\tilde{x}(u,\bar{v}) = \tilde{x}(\bar{u},\bar{v})$ for $\tilde{x} \in \tilde{X}$.



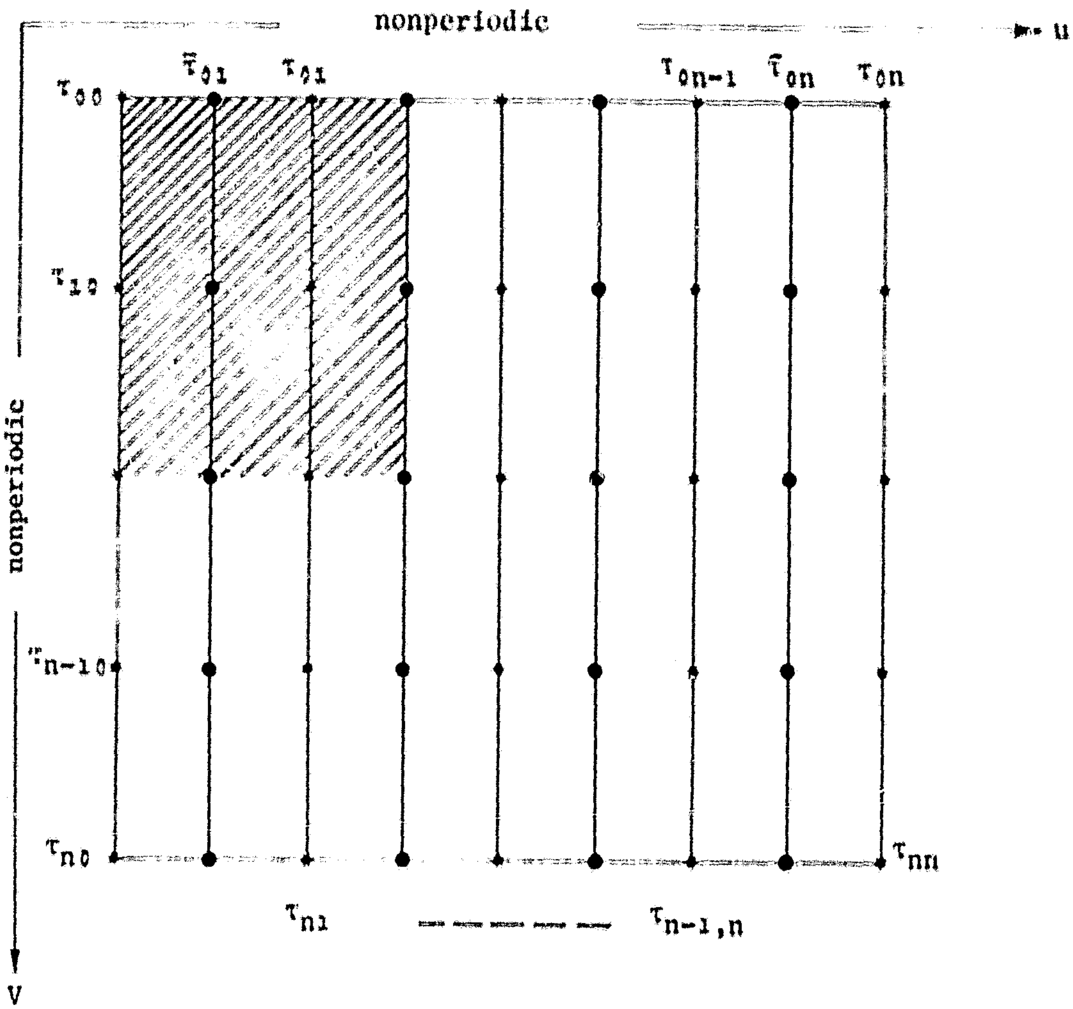
$$n(n + 1)$$

$$\bar{\tau}_{ij} \qquad \begin{array}{l} i = 0, \ldots, n \\ j = 1, \ldots n \end{array}$$

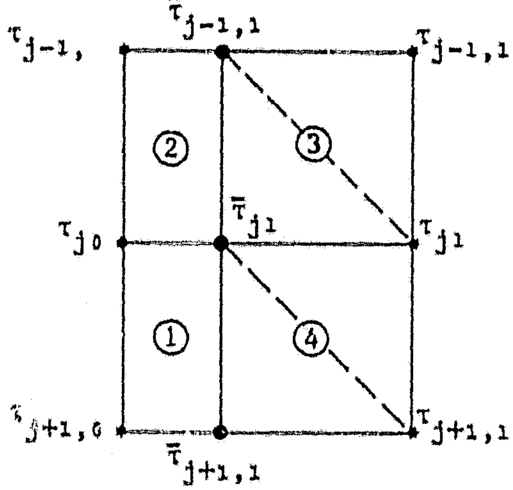
Therefore, $n(n + 1)$ is the dimension of the $\bar{\tau}_{ij}$ array.

$\bar{H}$ is defined by the $n(n+1) \times n(n+1)$ matrix as follows:

$$\bar{H} = \begin{Bmatrix} \frac{h}{2}(\tau_{00},\tau_{00}) & h(\tau_{00},\bar{\tau}_{01}) & \cdots & \frac{h}{2}(\tau_{00},\bar{\tau}_{n-1,n}) \\ \frac{h}{2}(\bar{\tau}_{01},\tau_{00}) & h(\bar{\tau}_{01},\bar{\tau}_{01}) & \cdots & \frac{h}{2}(\bar{\tau}_{01},\bar{\tau}_{n-1,n}) \\ \vdots & & & \\ \frac{h}{2}(\bar{\tau}_{0n},\tau_{00}) & h(\tau_{0n},\bar{\tau}_{01}) & \cdots & \frac{h}{2}(\bar{\tau}_{0n},\bar{\tau}_{n-1,n}) \\ \frac{h}{2}(\tau_{10},\tau_{00}) & h(\tau_{10},\bar{\tau}_{01}) & \cdots & \frac{h}{2}(\tau_{10},\bar{\tau}_{n-1,n}) \\ \vdots & & & \\ \frac{h}{2}(\bar{\tau}_{n-1,n},\tau_{00}) & h(\bar{\tau}_{n-1,n},\bar{\tau}_{01}) & \cdots & \frac{h}{2}(\bar{\tau}_{n-1,n},\bar{\tau}_{n-1,n}) \end{Bmatrix}$$



The coefficient of $\tilde{x}(\bar{\tau}_{j1})$ comes from rectangles 1, 2, 3, and 4 as shown.



Since

$$\iint\limits_{\square_{ij}} z(\bar{\tau}_{ij})[\tilde{x}(t) - \tilde{x}(\bar{\tau}_{ij})]dt = \frac{\Delta^2}{6} z(\bar{\tau}_{ij})[2\tilde{x}(\bar{\tau}_{i+1,j+1}) + \tilde{x}(\bar{\tau}_{i,j+1})$$

$$+ \tilde{x}(\bar{\tau}_{i+1,j}) - 4\tilde{x}(\bar{\tau}_{ij})]$$

and since

$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \leq \left[\omega_h\left(\frac{3}{4}\Delta\right) + \frac{2}{3}\omega_h(\sigma_2\Delta)\right] + \frac{3}{2}\Delta\|h\|$$

for the semiperiodic case, for this case it remains to determine upper estimates for the additional terms stemming from the nonperiodicity in the u-direction. It can readily be shown that the additional terms are bounded by

$$\Delta\left[\frac{7}{4}\|h\| + \frac{3}{4}\omega_h(\Delta)\right].$$

Therefore,

$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \leq \omega_h\left(\frac{3}{4}\Delta\right) + \frac{2}{3}\omega_h(\sqrt{2}\Delta) + \Delta\frac{3}{4}\omega_h(\Delta) + \frac{\Delta 13}{4}\|h\| ,$$

which completes the proof of condition A.

Conditions B and C

It can be shown, in a manner similar to that discussed in the Periodic Kernels section, that conditions B and C are satisfied.

CONCLUSIONS

The following system of equations

$$x(t_{ij}) - \lambda\sum_{k=1}^{n}\sum_{\ell=1}^{n} h(t_{ij};t_{k\ell})z(t_{k\ell})\Delta = y(t_{ij})$$

converges to the exact solution of the following two-dimensional Fredholm Integral equation of the second kind

$$x(r,s) - \lambda\int_0^1\int_0^1 h(r,s;t_1,t_2)x(t_1,t_2)dt_1\,dt_2 = y(r,s)$$

for the general cases of h being a nonperiodic, semiperiodic, and periodic function of r,s. The convergence is proved for three cases of h, namely, periodic, semiperiodic, and nonperiodic.

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